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RESEARCHES IN THE LUNAR THEORY,*

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WHEN we consider how we may best contribute to the advancement of this much-treated subject, we cannot fail to notice that the great majority of writers on it have had before them, as their ultimate aim, the construction of Tables: that is they have viewed the problem from the stand-point of practical astronomy rather than of mathematics. It is on this account that we find such a restricted choice of variables to express the position of the moon, and of parameters, in terms of which to express the coefficients of the periodic terms. Again, their object compelling them to go over the whole field, they have neglected to notice many minor points of great interest to the mathematician, simply because the knowledge of them was unnecessary for the formation of Tables. But the developments having now been carried extremely far, without completely satisfying all desires, one is led to ask whether such modifications cannot be made in the processes of integration, and such coördinates and parameters adopted, that a much nearer approach may be had to the law of the series, and, at the same time, their convergence augmented.

Now, as to choice of coördinates, it is known that, in the elliptic theory, the rectangular coördinates of a planet, relative to the central body, the axes being parallel to the axes of the ellipse described, can be developed, in terms of the time, in the following series,

$$x = a \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J_{\frac{ie}{2}}^{(i-1)} \cos ig,$$

$$y = b \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J_{\frac{ie}{2}}^{(i-1)} \sin ig,$$

a and b being the semi-axes of the ellipse, e the eccentricity, g the mean anomaly and, for positive values of i , the Besselian function (in Hansen's notation)

$$J_{\lambda}^{(i)} = \frac{\lambda^i}{1.2 \dots i} \left[1 - \frac{\lambda^2}{1.(i+1)} + \frac{\lambda^4}{1.2.(i+1)(i+2)} - \dots \right],$$

* Communicated to the National Academy of Sciences at the session of April, 1877.

while, for negative values of i , we have

$$J_{\lambda}^{(-i)} = J_{-\lambda}^{(i)},$$

and, for the special case of $i=0$, we put the indeterminate

$$\frac{1}{0} J_0^{(-1)} = -\frac{3}{2} e.$$

Here the law of the series is manifest, and the approximation can easily be carried as far as we wish. But the longitude and latitude, variables employed by nearly all the lunar theorists, are far from having such simple expressions; in fact, their coefficients cannot be expressed finitely in terms of Besselian functions. And if this is true in the elliptic theory, how much more likely is a similar thing to be true when the complexity of the problem is increased by the consideration of disturbing forces? We are then justified in thinking that the coefficients of the periodic terms in the development of rectangular or quasi-rectangular coördinates are less complex functions of their parameters than those of polar coördinates. There is also another advantage in employing coördinates of the former kind; the differential equations are expressed in purely algebraic forms; while, with the latter, circular functions immediately present themselves. For these reasons I have not hesitated to substitute rectangular for polar coördinates.

Again, as to parameters, all those who have given literal developments, Laplace setting the example, have used the parameter m , the ratio of the sidereal month to the sidereal year. But a slight examination, even, of the results obtained, ought to convince any one that this is a most unfortunate selection in regard to convergence. Yet nothing seems to render this parameter at all desirable, indeed, the ratio of the synodic month to the sidereal year would appear to be more naturally suggested as a parameter. Some instances of slow convergence with the parameter m may be given from Delaunay's Lunar Theory: the development of the principal part of the coefficient of the evection in longitude begins with the term $\frac{15}{4} me$, and ends with the term $\frac{413277465931033}{15288238080} m^8 e$; again, in the principal part of the coefficient of the inequality whose argument is the difference of the mean anomalies of the sun and moon, we find, at the beginning, the term $\frac{21}{4} mee'$, and, at the end, the term $\frac{1207454026843}{3538944} m^7 ee'$. It is probable that, by the adoption of some function of m as a parameter in place of this quantity, whose numerical

value, in the case of our moon, should not greatly exceed that of m , the foregoing large numerical coefficients might be very much diminished. And nothing compels us to use the same parameter throughout; one may be used in one class of inequalities, another in another, as may prove most advantageous. It is known what rapid convergence has been obtained in the series giving the values of logarithms, circular and elliptic functions, by simply adopting new parameters. Similar transformations, with like effects, are, perhaps, possible in the coefficients of the lunar inequalities. However, as far as my experience goes, no useful results are obtained by experimenting with the present known developments; in every case it seems the proper parameter must be deduced from *a priori* considerations furnished in the course of the integration.

With regard to the form of the differential equations to be employed, although Delaunay's method is very elegant, and, perhaps, as short as any, when one wishes to go over the whole ground of the lunar theory, it is subject to some disadvantages when the attention is restricted to a certain class of lunar inequalities. Thus, do we wish to get only the inequalities whose coefficients depend solely on m , we are yet compelled to develop the disturbing function R to all powers of e . Again, the method of integrating by undetermined coefficients is most likely to give us the nearest approach to the law of the series; and, in this method, it is as easy to integrate a differential equation of the second order as one of the first, while the labor is increased by augmenting the number of variables and equations. But Delaunay's method doubles the number of variables in order that the differential equations may be all of the first order. Hence, in this disquisition, I have preferred to use the equations expressed in terms of the coördinates, rather than those in terms of the elements; and, in general, always to diminish the number of unknown quantities and equations by augmenting the order of the latter. In this way the labor of making a preliminary development of R in terms of the elliptic elements is avoided.

In the present memoir I propose, dividing the periodic developments of the lunar coördinates into classes of terms, after the manner of Euler in his last *Lunar Theory*,* to treat the following five classes of inequalities:—

1. Those which depend only on the ratio of the mean motions of the sun and moon.

* *Theoria Motuum Lunæ, nova methodo pertractata.* Petropoli, 1772.

2. Those which are proportional to the lunar eccentricity.
3. Those which are proportional to the sine of the lunar inclination.
4. Those which are proportional to the solar eccentricity.
5. Those which are proportional to the solar parallax.

A general method will also be given by which these investigations may be extended so as to cover the whole ground of the lunar theory. My methods have the advantage, which is not possessed by Delaunay's, that they adapt themselves equally to a special numerical computation of the coefficients, or to a general literal development. The application of both procedures will be given in each of the just mentioned five classes of inequalities, so that it will be possible to compare them.

I regret that, on account of the difficulty of the subject and the length of the investigations it seems to require, I have been obliged to pass over the important questions of the limits between which the series are convergent, and of the determination of superior limits to the errors committed in stopping short at definite points. There cannot be a reasonable doubt that, in all cases, where we are compelled to employ infinite series in the solution of a problem, analysis is capable of being perfected to the point of showing us within what limits our solution is legitimate, and also of giving us a limit which its error cannot surpass. When the coördinates are developed in ascending powers of the time, or in ascending powers of a parameter attached as a multiplier to the disturbing forces, certain investigations of Cauchy afford us the means of replying to these questions. But when, for powers of the time, are substituted circular functions of it, and the coefficients of these are expanded in powers and products of certain parameters produced from the combination of the masses with certain of the arbitrary constants introduced by integration, it does not appear that anything in the writings of Cauchy will help us to the conditions of convergence.

CHAPTER I.

Differential Equations.—Properties of motion derived from Jacobi's integral.

We set aside the action of the planets and the influence of the figures of the sun, earth and moon, together with the action of the last upon the

sun, as also the product of the solar disturbing force on the moon by the small fraction obtained from dividing the mass of the earth by the mass of the sun. These are the same restrictions as those which Delaunay has imposed on his Lunar Theory contained in Vols. XXVIII and XXIX of the Memoirs of the Paris Academy of Sciences. The motion of the sun, about the earth, is then in accordance with the elliptic theory, and the ecliptic is a fixed plane.

Let us take a system of rectangular axes, having its origin at the centre of gravity of the earth, the axis of x being constantly directed toward the centre of the sun, the axis of y toward a point in the ecliptic 90° in advance of the sun, and the axis of z perpendicular to the ecliptic. In addition, we adopt the following notation:—

r' = the distance of the sun from the earth;

λ' = the sun's longitude;

μ = the sum of the masses of the earth and moon, measured by the velocity these masses produce by their action, in a unit of time, and at the unit of distance;

m' = the mass of the sun, measured in the same way;

n' = the mean angular velocity of the sun about the earth;

a' = the sun's mean distance from the earth.

In accordance with one of the above-mentioned restrictions we have the equation:—

$$m' = n'^2 a'^3$$

The axes of x and y having a velocity of rotation in their plane, equal to $\frac{d\lambda'}{dt}$, it is evident that the square of the velocity of the moon, relative to the earth's centre, has for expression, in terms of the adopted coördinates,

$$\begin{aligned} 2 T &= \left[\frac{dx}{dt} - y \frac{d\lambda'}{dt} \right]^2 + \left[\frac{dy}{dt} + x \frac{d\lambda'}{dt} \right]^2 + \frac{dz^2}{dt^2} \\ &= \frac{dx^2 + dy^2 + dz^2}{dt^2} + 2 \frac{d\lambda'}{dt} \frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2} (x^2 + y^2). \end{aligned}$$

The potential function, in terms of the same coordinates, is

$$\Omega = \frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + \frac{n'^2 a'^3}{\sqrt{[(r' - x)^2 + y^2 + z^2]}} - \frac{n'^2 a'^3}{r'^2} x.$$

If the second radical in this expression is expanded in a series proceeding according to descending powers of r' , and the first term $\frac{n'^2 a'^3}{r'}$ omitted, since it

disappears in all differentiations with respect to the moon's coördinates, the following expression is obtained

$$\begin{aligned}\Omega = & \frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + n^2 \frac{a'^3}{r^3} \left[x^2 - \frac{1}{2} (y^2 + z^2) \right] \\ & + \frac{n'^2}{a'} \frac{a'^4}{r^4} \left[x^3 - \frac{3}{2} x (y^2 + z^2) \right] \\ & + \frac{n'^2}{a'^2} \frac{a'^5}{r^5} \left[x^4 - 3x^2 (y^2 + z^2) + \frac{3}{8} (y^2 + z^2)^2 \right] \\ & + \frac{n'^2}{a'^3} \frac{a'^6}{r^6} \left[x^5 - 5x^3 (y^2 + z^2) + \frac{15}{8} x (y^2 + z^2)^2 \right] \\ & + \dots\end{aligned}$$

Since the differential equations of motion are of the form

$$\frac{d}{dt} \cdot \frac{dT}{d\phi} - \frac{dT}{d\phi} = \frac{d\Omega}{d\phi},$$

ϕ denoting, in succession, each of the variables which define the position of the moon, it is plain that the term

$$\frac{1}{2} \frac{d\lambda'^2}{dt^2} (x^2 + y^2)$$

may be removed from T and added to Ω ; and these modified quantities may be denoted by the symbols T' and Ω' . Then these equations may be written thus:

$$\begin{aligned}\frac{d^2x}{dt^2} - 2 \frac{d\lambda'}{dt} \frac{dy}{dt} - \frac{d^2\lambda'}{dt^2} y &= \frac{d\Omega'}{dx}, \\ \frac{d^2y}{dt^2} + 2 \frac{d\lambda'}{dt} \frac{dx}{dt} + \frac{d^2\lambda'}{dt^2} x &= \frac{d\Omega'}{dy}, \\ \frac{d^2z}{dt^2} &= \frac{d\Omega'}{dz}.\end{aligned}$$

When we wish to restrict our attention to the lunar inequalities which are independent of the solar parallax, all the terms, in the last expression of Ω , which are divided by r^4 , r^5 , r^6 , &c., may be omitted. In this case it will be seen that all the terms, introduced into the differential equations by the solar action, are linear in form, with variable, but known coefficients, since $\frac{d\lambda'}{dt}$, $\frac{d^2\lambda'}{dt^2}$ and $\frac{a'^3}{r^3}$ are known functions of t .

When all the inequalities, involving the solar eccentricity, are neglected, the equations admit an integral in finite terms. For, in this case, we have

$$\frac{d\lambda'}{dt} = n', \quad \frac{d^2\lambda'}{dt^2} = 0, \quad r' = a',$$

and Ω' does not explicitly contain t ; hence, multiplying the equations severally by the factors dx , dy , and dz , and adding the products, both members of the resulting equation are exact differentials. Integrating this equation, we have

$$\frac{dx^2 + dy^2 + dz^2}{2dt^2} = \Omega' + \text{a constant.}$$

This integral equation appears to have been first obtained by Jacobi.* As it will be frequently referred to in what follows, I shall take the liberty of calling it Jacobi's integral.

If we take two imaginary variables

$$u = x + \sqrt{(y^2 + z^2)} \sqrt{-1}, \\ s = x - \sqrt{(y^2 + z^2)} \sqrt{-1},$$

Ω has the following simple expression, being a function of two variables only,

$$\Omega = \frac{\mu}{\sqrt{us}} + \frac{n'^2 a'^3}{\sqrt{(r' - u)} \sqrt{(r' - s)}} - \frac{n'^2 a'^3}{2r'^2} (u + s).$$

If this is expanded in descending powers of r' , and, as before, the term $\frac{n'^2 a'^3}{r'}$ omitted,

$$\begin{aligned} \Omega = & \frac{\mu}{\sqrt{us}} + n'^2 \frac{a'^3}{r'^3} \left[\frac{3}{8} u^2 + \frac{1}{4} us + \frac{3}{8} s^2 \right] \\ & + \frac{n'^2 a'^4}{a' r'^4} \left[\frac{5}{16} u^3 + \frac{3}{16} u^2 s + \frac{3}{16} us^2 + \frac{5}{16} s^3 \right] \\ & + \frac{n'^2 a'^5}{a'^2 r'^5} \left[\frac{35}{128} u^4 + \frac{5}{32} u^3 s + \frac{9}{64} u^2 s^2 + \frac{5}{32} us^3 + \frac{35}{128} s^4 \right] \\ & + \dots \end{aligned}$$

The additional variable, necessary to complete the definition of the moon's position, may be so taken that the expression of T may be simplified as much as possible. This expression may be written

$$2 T = \frac{duds}{dt^2} - 4 \frac{(ydz - zdy)^2}{(u - s)^2 dt^2} + 2 \frac{d\lambda'}{dt} \frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2} (us - z^2).$$

* *Comptes Rendus de l'Académie des Sciences de Paris.* Tom. iii, p. 59.

There does not seem to be any function of x , y and z , which, adopted as a new variable to accompany u and s , would reduce this to a very simple form. However, when we are engaged in determining the inequalities independent of the inclination of the lunar orbit, this transformation will be useful to us. For, in this case, $z = 0$, and the values of u and s become

$$\begin{aligned} u &= x + y \sqrt{-1}, \\ s &= x - y \sqrt{-1}, \end{aligned}$$

and T is given by the equation

$$2 T = \frac{duds}{dt^2} - \frac{d\lambda'}{dt} \frac{uds - sdu}{dt} + \frac{d\lambda'^2}{dt^2} us.$$

Although Ω is expressed most simply by the systems of coordinates we have just employed, the integration of the differential equations will be easier, if we suppose that the axes of x and y have a constant instead of a variable velocity of rotation, the axis of x being made to pass through the mean position of the sun instead of the true. To obtain the expression for T correspondent to this supposition, it is necessary only to write n' for $\frac{d\lambda'}{dt}$ in the former values. As for Ω , it can be written thus

$$\Omega = \frac{\mu}{r} + \frac{n^2 a^3}{[r'^2 - 2r'S + r^2]^{\frac{1}{2}}} - \frac{n^2 a^3}{r'^2} S,$$

where

$r^2 = x^2 + y^2 + z^2$ = the square of the moon's radius vector ;

$S = x \cos v + y \sin v$;

v = the solar equation of the centre.

This function being expanded in a series of descending powers of r' , as before, we have

$$\begin{aligned} \Omega' &= \frac{\mu}{r} + \frac{1}{2} n'^2 (x^2 + y^2) \\ &\quad + n'^2 \frac{a^3}{r'^3} \left[\frac{3}{2} S^2 - \frac{1}{2} r'^2 \right] \\ &\quad + \frac{n'^2}{a'} \frac{a'^4}{r'^4} \left[\frac{5}{2} S^3 - \frac{3}{2} r'^2 S \right] \\ &\quad + \frac{n'^2}{a'^2} \frac{a'^5}{r'^5} \left[\frac{35}{8} S^4 - \frac{15}{4} r'^2 S^2 + \frac{3}{8} r'^4 \right] \\ &\quad + \frac{n'^2}{a'^3} \frac{a'^6}{r'^6} \left[\frac{63}{8} S^5 - \frac{35}{4} r'^2 S^3 + \frac{15}{8} r'^4 S \right] \\ &\quad + \dots \end{aligned}$$

And the corresponding differential equations are

$$\begin{aligned}\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} &= \frac{d\Omega'}{dx}, \\ \frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} &= \frac{d\Omega'}{dy}, \\ \frac{d^2z}{dt^2} &= \frac{d\Omega'}{dz},\end{aligned}$$

Thus much in reference to the equations under as general a form as we shall have occasion for in the present disquisition. We shall now suppose that they are reduced to as restricted a form as is possible without their becoming the equations of the elliptic theory; that is, we shall assume that the solar parallax and eccentricity and the lunar inclination vanish. With these simplifications, in the first system of coördinates,

$$T' = \frac{dx^2 + dy^2}{2dt^2} + n' \frac{xdy - ydx}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{(x^2 + y^2)}} + \frac{3}{2} n'^2 x^2;$$

and, in the second,

$$T' = \frac{duds}{2dt^2} - \frac{n'}{2} \frac{uds - sdu}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{us}} + \frac{3}{8} n'^2 (u + s)^2.$$

And the differential equations, correspondent, are, in the first case,

$$\begin{aligned}\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2 \right] x &= 0, \\ \frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} + \frac{\mu}{r^3} y &= 0,\end{aligned}$$

and, in the second,

$$\begin{aligned}\frac{d^2u}{dt^2} - 2n' \frac{du}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}} u - \frac{3}{2} n'^2 (u + s) &= 0, \\ \frac{d^2s}{dt^2} + 2n' \frac{ds}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}} s - \frac{3}{2} n'^2 (u + s) &= 0.\end{aligned}$$

The Jacobian integral has severally the expressions

$$\begin{aligned}\frac{dx^2 + dy^2}{2dt^2} &= \frac{\mu}{r} + \frac{3}{2} n'^2 x^2 - C, \\ \frac{duds}{2dt^2} &= \frac{\mu}{\sqrt{us}} + \frac{3}{8} n'^2 (u + s)^2 - C.\end{aligned}$$

The terms $-2n' \frac{dy}{dt}$, $2n' \frac{dx}{dt}$, &c., have been introduced into the equations by making the axes of coördinates movable; but since the putting of $n' = 0$ makes the solar disturbing force vanish, there is no inconsistency in attributing them to the solar action. Then, in the case of the vanishing of this action, we have the equations of ordinary elliptic motion

$$\begin{aligned}\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x &= 0, \\ \frac{d^2y}{dt^2} + \frac{\mu}{r^3}y &= 0.\end{aligned}$$

Thus, in the restricted case we consider, all the terms, added to the differential equations of motion by the solar action, are linear in form and have constant coefficients. This, and the circumstance that t does not explicitly appear in the equations, are two advantages which are due to the particular selection of the variables x and y . If $\frac{\mu}{r^3}$ were constant, the equations would be linear with constant coefficients and easily integrable.

The constants μ and n' can be made to disappear from the differential equations, if, instead of leaving the units of length and time arbitrary, we assume them so that $\mu = 1$, and $n' = 1$. The new unit of length, will then be equal to $\sqrt[3]{\frac{\mu}{n'^2}}$ units of the previous measurement. The equations, thus simplified, are

$$\begin{aligned}\frac{d^2x}{dt^2} - 2 \frac{dy}{dt} + \left[\frac{1}{r^3} - 3 \right] x &= 0, \\ \frac{d^2y}{dt^2} + 2 \frac{dx}{dt} + \frac{1}{r^3} y &= 0,\end{aligned}$$

with their integral

$$\frac{dx^2 + dy^2}{dt^2} = \frac{2}{r} + 3x^2 - 2C.$$

It will be perceived that, in this way, we make the differential equations absolutely the same for all cases of the satellite problem.

Let us put $\rho = \frac{dx}{dt}$, then

$$\begin{aligned}\frac{dy}{dt} &= \left[\frac{2}{r} + 3x^2 - 2C - \rho^2 \right]^{\frac{1}{2}}, \\ \frac{d\rho}{dt} &= 2 \left[\frac{2}{r} + 3x^2 - 2C - \rho^2 \right]^{\frac{1}{2}} - \left[\frac{1}{r^3} - 3 \right] x.\end{aligned}$$

Or, by making y the independent variable,

$$\frac{dx}{dy} = \frac{\rho}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}},$$

$$\frac{d\rho}{dy} = 2 - \frac{\left[\frac{1}{r^3} - 3\right] x}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}}.$$

The problem is then reduced to the integration of two differential equations of the first order. Were this accomplished, and ρ eliminated from the two integral equations, we should have the equation of the orbit. If we put

$$W = 2x + \left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}},$$

the differential equations can be written in the canonical form,

$$\frac{dx}{dy} = -\frac{dW}{d\rho},$$

$$\frac{d\rho}{dy} = \frac{dW}{dx}.$$

It may be worth while to notice also the single partial differential equation, to the integration of which our problem can be reduced. Returning to the arbitrary linear and temporal units, and, for convenience, reversing the sign of C , if a function of x and y can be found satisfying the partial differential equation

$$\left[\frac{dV}{dx} + n'y\right]^2 + \left[\frac{dV}{dy} - n'x\right]^2 = \frac{2\mu}{\sqrt{(x^2 + y^2)}} + 3n'^2x^2 + 2C,$$

and involving a single arbitrary constant h , distinct from that which can be joined to it by addition, the intermediate integrals of the problem will be

$$\frac{dx}{dt} = \frac{dV}{dx} + n'y, \quad \frac{dy}{dt} = \frac{dV}{dy} - n'x,$$

and the final integrals

$$\frac{dV}{dh} = \alpha, \quad \frac{dV}{dC} = t + c,$$

α and c being two additional arbitrary constants. The truth of this will be evident if we differentiate the four integral equations with respect to t and compare severally the results with the partial differential coefficients of the partial differential equation with respect to x , y , h and C .

Although, in this manner, the problem seems reduced to its briefest terms, yet, when we essay to solve it, setting out with this partial differential equation, we are led to more complex expressions than would be expected. It would be advisable, in this method of proceeding, to substitute polar for rectangular coördinates, or to put

$$x = r \cos \phi, \quad y = r \sin \phi.$$

The partial differential equation, thus transformed, is

$$\frac{dV^2}{dr^2} + \left[\frac{1}{r} \frac{dV}{d\phi} - n'r \right]^2 = \frac{2\mu}{r} + \frac{3}{2} n'^2 r^2 + 2C + \frac{3}{2} n'^2 r^2 \cos 2\phi.$$

This would have to be integrated by successive approximations, and it is found that this method, which, at first sight, seems likely to afford a briefer solution of the problem, because but one unknown quantity was to be determined, and this free from the variable t , and involving only half of the number of arbitrary constants introduced by integration, when developed, leads to as complex operations as the older methods, and, moreover, has the disadvantage of giving results which need prolix transformations before the coördinates can be exhibited in terms of the time.

Although we shall make no use of the equations in terms of polar coördinates, they may be given here, for the sake of some special properties they possess in this form. They are

$$\begin{aligned} r \frac{d^2 r}{dt^2} - r^2 \frac{d\phi^2}{dt^2} - 2n'r^2 \frac{d\phi}{dt} + \frac{\mu}{r} - 3n'^2 r^2 \cos^2 \phi &= 0, \\ \frac{d}{dt} \left[r^2 \left(\frac{d\phi}{dt} + n' \right) \right] + \frac{3}{2} n'^2 r^2 \sin 2\phi &= 0, \end{aligned}$$

with their integral

$$\frac{dr^2 + r^2 d\phi^2}{dt^2} = \frac{2\mu}{r} + 3n'^2 r^2 \cos^2 \phi - 2C.$$

By dividing the second of the differential equations by r^2 , the variables are separated, and λ denoting the longitude of the moon, we have

$$r = \frac{K}{\sqrt{\frac{d\lambda}{dt}}} \varepsilon - \frac{3}{2} n'^2 \int \frac{\sin 2(\lambda - \lambda')}{\frac{d\lambda}{dt}} dt$$

K being a constant. Thus, after the longitude is determined in terms of t , the radius vector is obtained by a quadrature. But it can also be found, without the necessity of an integration, by dividing the integral by r^2 and

then eliminating the term $\frac{1}{r^2} \frac{dr^2}{dt^2}$ by means of its value derived from the second differential equation; in this way we get

$$\frac{\mu}{r^3} - \frac{C}{r^2} = \frac{1}{8} \left[\frac{\frac{d^2\phi}{dt^2} + \frac{3}{2} n'^2 \sin 2\phi}{\frac{d\phi}{dt} + n'} \right]^2 + \frac{1}{2} \frac{d\phi^2}{dt^2} - \frac{3}{2} n'^2 \cos^2 \phi.$$

As we desire to make constant numerical application of the general theory, established in what follows, to the particular case of the moon, we delay here, for a moment, to obtain the numerical values of the three constants μ , n' and C . The value of μ may be derived either from the observed value of the constant of lunar parallax combined with the mean angular motion of the moon, or from the intensity of gravity at the earth's surface and the ratio of the moon's mass to that of the earth. We will adopt the latter procedure. The value of gravity at the equator, $g = 9.779741$ metres, the unit of time being the mean solar second. We propose, however, to take the mean solar day as the unit of time, and the equatorial radius of the earth as the linear unit. This number must then be multiplied by $\frac{86400^2}{6377397.15}$, (6377397.15 metres is Bessel's value of the equatorial radius.) Moreover, the theory of the earth's figure shows that, in order to obtain the attractive force of the earth's mass, considered as concentrated at its centre of gravity, a second multiplication must be made by the factor 1.001818356. With our units then this force is represented by the number 11468.338: and the moon's mass being taken at $\frac{1}{81.52277}$ of the earth's, her attractive force is represented by the number 140.676. Consequently

$$\mu = 11609.014.$$

The sidereal mean motion of the sun in a Julian year is 1295977''.41516, whence

$$n' = 0.017202124.$$

The value of C might be obtained from the observed values of the moon's coördinates and their rates of variation at any time. However, as the eccentricity of the earth's orbit is not zero, C obtained in this manner would be found to undergo slight variations. The mean of all the values obtained in a long series of observations might be adopted as the proper value of this

quantity when regarded as constant. But it is much easier to derive it approximately from the series

$2C = (\mu n)^{\frac{2}{3}} [1 + 2m - \frac{5}{2} m^2 - m^3 - \frac{1319}{288} m^4 - \frac{67}{144} m^5 - \frac{2879}{1296} m^6 - \frac{1321}{1296} m^7]$,
which will be established in the following chapter. Here n denotes the moon's sidereal mean motion, and m is put for $\frac{n'}{n - n'}$. In this formula the

terms which involve the squares of the lunar eccentricity and inclination and of the solar parallax are neglected; this, however, is not of great moment, as, being multiplied by at least m^2 , they are of the fourth order with respect to smallness. The observations give $n = 0.22997085$, hence

$$C = 111.18883.$$

If it is proposed to assume the units of time and length so that μ and n' may both be unity, it will be found that the first is equal to 58.13236 mean solar days, and the second to 339.7898 equatorial radii of the earth. The corresponding value of C is 3.254440.

Let us now notice some of the properties of motion which can be derived from Jacobi's integral. This integral gives the square of the velocity relative to the moving axes of coördinates; and, as this square is necessarily positive, the putting it equal to zero gives the equation of the surface which separates those portions of space, in which the velocity is real, from those in which it is imaginary. This equation is, in its most general form,

$$\frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + \frac{n'^2 a^3}{\sqrt{[(a' - x)^2 + y^2 + z^2]}} = C + \frac{3}{2} n'^2 a'^2 - \frac{n'^2}{2} [(a' - x)^2 + y^2],$$

which is seen to be of the 16th degree. As y and z enter it only in even powers, the surface is symmetrically situated with respect to the planes of xy and xz . The left member is necessarily positive, (the folds of the surface, for which either or both the radicals receive negative values, are excluded from consideration), hence the surface is inclosed within the cylinder whose axis passes through the centre of the sun perpendicularly to the ecliptic, and whose trace on this plane is a circle of the radius

$$a' \sqrt{\left(3 + \frac{2C}{n'^2 a'^2}\right)}.$$

As, in general, the second term of the quantity, under the radical sign, is much smaller than the first, this radius is, quite approximately $\sqrt{3} a'$. Thus, in the case of our moon, assuming $\frac{1}{a'} = \sin 8''.848$, we have this radius =

$\sqrt{3.001383} a'$. It is evident that, for all points without this cylinder, the velocity is real; and as, for large values of z , whether positive or negative, the left member of the equation becomes very small, it is plain that the cylinder is asymptotic to the surface. Every right line, perpendicular to the ecliptic, intersects the surface not more than twice, at equal distances from this plane, once above and once below.

Let us now find the trace of the surface on the plane of xy . Putting ρ for the distance of a point on this trace from the centre of the sun,

$$\rho^2 = (a' - x)^2 + y^2.$$

and it is evident that the cubic equation,

$$\rho^3 = a'^2 \left(3 + \frac{2C}{n'^2 a'^2} \right) \rho - 2a'^3,$$

will give the limits between which the values of ρ can oscillate. If C is negative, this equation has but one real root which is negative; consequently, in this case, the surface has no intersection with the plane of xy . But, in all the satellite systems we know, C is positive, and this condition is probably necessary to insure stability. Hence we shall restrict our attention to the case where C is positive. Then all the roots of the equation are real, and two are positive. It is between the latter roots that ρ must always be found. To compute them, we derive the auxillary angle θ from the formula

$$\sin \theta = \left[1 + \frac{2}{3} \frac{C}{n'^2 a'^2} \right]^{-\frac{3}{2}},$$

or, since θ differs but little from 90° , with more readiness from

$$\cos^2 \theta = \frac{2 \frac{C}{n'^2 a'^2} \left[1 + \frac{2}{3} \frac{C}{n'^2 a'^2} + \frac{4}{27} \frac{C^2}{n'^4 a'^4} \right]}{\left[1 + \frac{2}{3} \frac{C}{n'^2 a'^2} \right]^3},$$

or, as $\frac{C}{n'^2 a'^2}$ is a small quantity, with sufficient approximation from

$$\cos \theta = \frac{\sqrt{2 \frac{C}{n'^2 a'^2}}}{1 + \frac{2}{3} \frac{C}{n'^2 a'^2}}.$$

The two roots are then

$$\begin{aligned} \rho_1 &= 2a' \sqrt{1 + \frac{2}{3} \frac{C}{n'^2 a'^2}} \sin \frac{\theta}{3}, \\ \rho_2 &= 2a' \sqrt{1 + \frac{2}{3} \frac{C}{n'^2 a'^2}} \sin \left(60^\circ - \frac{\theta}{3} \right) \end{aligned}$$

The trace of the surface on the plane of xy is then wholly comprised in the annular space between the two circles described from the centre of the sun as centre with the radii ρ_1 and ρ_2 . Moreover, as in most satellite systems we have $\frac{\mu}{n'^2 a'^3}$ equal to a very small fraction, (for our moon $\frac{\mu}{n'^2 a'^3} = \frac{1}{322930.2}$), it is plain that, for points whose distance from the earth is comparable with their distance from the sun, the trace is approximately coincident with these circles. For the term $\frac{\mu}{r}$, in the equation, may then be neglected in comparison with the other terms.

In the case of our moon there is found

$$\theta = 87^\circ 52' 11''.53.$$

and hence

$$\rho_1 = 22815.15. \qquad \rho_2 = 23816.09,$$

and, if r and ρ are regarded as the variables defining the position of a point in the plane xy , the following table gives some corresponding values of these quantities, for each of the two branches of the trace approximating severally to the two circles.

$r.$	$\rho.$	$r.$	$\rho.$
433.3257	22878.69	439.7922	23751.81
450	22876.17	450	23753.37
500	22869.68	500	23760.04
600	22860.13	600	23769.85
1000	22841.59	1000	23788.87
10000	22817.70	10000	23813.43
46127.70	22815.68	47127.55	23815.53

The first and last values correspond to the four points where the curves intersect the axis of x on the hither and thither side of the sun. It will be seen that the approximation of the branches to the circles is quite close, except in the vicinity of the earth, where there is a slight protruding away from them.

In addition to these two branches of the trace, there is, in the case where C exceeds a certain limit, a third closed one about the origin much smaller than the former. As the coördinates of points in this branch are small fractions of a' , its equation may be written, quite approximately,

$$\frac{\mu}{r} = C - \frac{3}{2} n'^2 x^2.$$

It intersects the axis of y at a distance from the origin very nearly

$$y_0 = \frac{\mu}{C},$$

and the axis of x at points whose coördinates are the smallest (without regard to sign) roots of the equations

$$\begin{aligned} \frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} &= C + \frac{3}{2} n'^2 a'^2 - \frac{1}{2} n'^2 (a' - x)^2, \\ -\frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} &= C + \frac{3}{2} n'^2 a'^2 - \frac{1}{2} n'^2 (a' - x)^2 \end{aligned}$$

For the moon these quantities have the values

$$y_0 = 104.408, \quad x_1 = -109.655, \quad x_2 = +109.694.$$

This branch then does not differ much from a circle having its centre at the origin, more closely it approximates to the ellipse whose major axis $= x_2 - x_1$, and minor axis $= 2y_0$.

The value of the coördinate z , for the single intersection of the surface with the axis of z above the plane of xy , is given by the single positive root of the equation

$$\frac{\mu}{z} + \frac{n'^2 a'^3}{\sqrt{(a'^2 + z^2)}} = C + n'^2 a'^2.$$

For the moon the numerical value of this root is

$$z_0 = 102.956.$$

The intersection of the surface with the perpendicular to the plane of xy passing through the centre of the sun is, in like manner, given by the equation

$$\frac{\mu}{\sqrt{(a'^2 + z^2)}} + \frac{n'^2 a'^3}{z} = C + \frac{3}{2} n'^2 a'^2,$$

having but a single positive root, which is nearly

$$z_0 = \frac{\frac{2}{3} a'}{1 + \frac{3}{2} \frac{C}{n'^2 a'^2}},$$

or, with less exactitude,

$$z_0 = \frac{2}{3} a'.$$

From these investigations it is possible to get a tolerably clear idea of the form of this surface. When C exceeds a certain limit, it consists of three separate folds. The first being quite small, relatively to the other two, is closed, surrounds the earth and somewhat resembles an ellipsoid whose axes have been given above. The second is also closed, but surrounds the sun,

and has approximately the form of an ellipsoid of revolution, the semiaxis in the plane of the ecliptic being somewhat less than a' , and the semiaxis of revolution perpendicular to the ecliptic and passing through the sun being about two-thirds of this. This fold has a protuberance in the portion neighboring the earth. The third fold is not closed, but is asymptotic to the cylinder mentioned at the beginning of the investigation of the surface. Like the second, it also is nearly of revolution about an axis passing through the centre of the sun and perpendicular to the ecliptic. The radius of its trace on the ecliptic is about as much greater than a' , as the radius of the trace of the second fold falls short of that quantity. The fold has a protuberance in the portion neighboring the earth, and which projects towards this body. The whole fold resembles a cylinder bent inwards in a zone neighboring the ecliptic.

What modifications take place in these folds when the constants involved in the equation of the surface are made to vary, will be clearly seen from the following exposition. Let us, for brevity, put

$$h = 3 + 2 \frac{C}{n'^2 a'^2},$$

and, for the moment, adopt a' , the distance of the earth from the sun, as the linear unit, and transfer the origin to the centre of the sun, and moreover put

$$\gamma = \frac{\mu}{n'^2 a'^3}.$$

Then the intersections of the surface, with the axis of x , will be given by the two roots of the equation

$$x^4 - x^3 - hx^2 + (h + 2 - 2\gamma)x - 2 = 0,$$

which lie between the limits 0 and 1; by the two roots of

$$x^4 - x^3 - hx^2 + (h + 2 + 2\gamma)x - 2 = 0,$$

which lie between 1 and $\sqrt{h}$; and by the two roots of

$$x^4 - x^3 - hx^2 + (h - 2 - 2\gamma)x + 2 = 0,$$

which lie between 0 and $-\sqrt{h}$.

Hence, if C diminishes so much that the first of these three equations has the two roots, lying between the mentioned limits, equal, the first fold will have a contact with the second fold; and if C fall below this limit, the roots become imaginary, and the two folds become one. Again, if C is diminished to the limit where the second equation has the mentioned pair of roots equal, the first fold will have a contact with the third; and when C is

less than this, these two folds form but one. And when C is less than both these limits, there will be but one fold to the surface.

In the spaces inclosed by the first and second folds the velocity, relative to the moving axes of coördinates, is real; but, in the space lying between these folds and the third fold, it is imaginary; without the third fold it is again real. Thus, in those cases, where C and γ have such values that the three folds exist, if the body, whose motion is considered, is found at any time within the first fold, it must forever remain within it, and its radius vector will have a superior limit. If it be found within the second fold, the same thing is true, but the radius vector will have an inferior as well as a superior limit. And if it be found without the third fold, it must forever remain without, and its radius vector will have an inferior limit.

Applying this theory to our satellite, we see that it is actually within the first fold, and consequently must always remain there, and its distance from the earth can never exceed 109.694 equatorial radii. Thus, the eccentricity of the earth's orbit being neglected, we have a rigorous demonstration of a superior limit to the radius vector of the moon.

In the cases, where C and γ have such values that the surface forms but one fold, Jacobi's integral does not afford any limits to the radius vector.

When we neglect the solar parallax and the lunar inclination, the preceding investigation is reduced to much simpler terms. The surface then degenerates into a plane curve, whose equation, of the sixth degree, is

$$\frac{\mu}{r} = C - \frac{3}{2} n^2 x^2.$$

It is evidently symmetrical with respect to both axes of coördinates, and is contained between the two right lines, whose equations are

$$x = \pm \sqrt{\frac{2C}{3n^2}},$$

and which are asymptotic to it. It intersects the axis of y , at two points, whose coordinates are

$$y = \pm \frac{\mu}{C}.$$

The cubic equation,

$$r^3 - \frac{2C}{3n^2} r + \frac{2\mu}{3n^2} = 0$$

gives the values of r , for which the curves intersect the axis of x . If

$$(2C)^{\frac{3}{2}} > 9\mu n',$$

this equation has two real roots between the limits 0 and $+\sqrt{\frac{2C}{3n'^2}}$. If

$$(2C)^{\frac{3}{2}} = 9\mu n',$$

these roots become equal. And if

$$(2C)^{\frac{3}{2}} < 9\mu n',$$

there are no real roots between these limits, and the curve has no intersection with the axis of x . The figures below exhibit the three varieties of this curve.

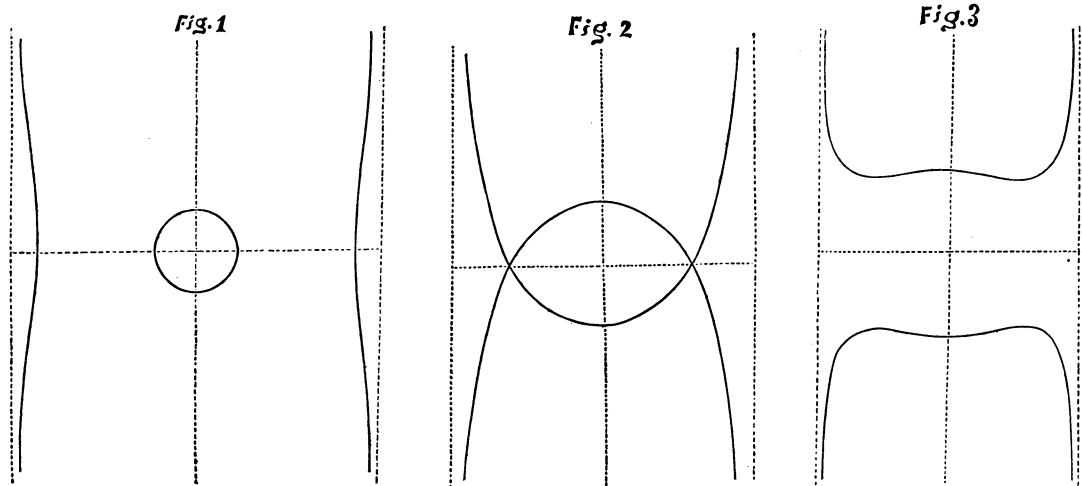


Fig. 1 represents the form of the curve in the case of our moon. In Fig. 2 we see that the small oval of Fig. 1 has enlarged and elongated itself so as to touch the two infinite branches; while, in Fig. 3, it has disappeared, the portions of the curve, lying on either side of the axis of x , having lifted themselves away from it, and the angles having become rounded off. In Fig. 1, the velocity is real within the oval, and also without the infinite branches, but it is imaginary in the portion of the plane lying between the oval and these branches. Hence, if the body be found, at any time, within the oval, it cannot escape thence, and its radius vector will have a superior limit; and, if it be found in one of the spaces on the concave side of the infinite branches, it cannot remove to the other, and its radius vector will have an inferior limit.

In the case represented in Fig. 2, the same things are true, but it seems as if the body might escape from the oval to the infinite spaces, or vice versa, at the points where the curve intersects the axis of x . However, at these points, the force, no less than the velocity, is reduced to zero. For the distance of these points from the origin is the positive root of the equation

$$3r^2 - \frac{2C}{3n'^2} = 0,$$

or

$$\frac{\sqrt{2C}}{3n'} = \frac{\sqrt[3]{9\mu n'}}{3n'},$$

and this value is the same as that given by the equation

$$\frac{\mu}{r^3} - 3n'^2 = 0.$$

In consequence the forces vanish at these two points, and thus we have two particular solutions of our differential equations.*

In the case represented in Fig. 3, the integral does not afford any superior or inferior limit to the radius vector.

The surface, or, in the more simple case, the plane curve, we have discussed, is the locus of zero velocity; and the surface or plane curve, upon which the velocity has a definite value, is precisely of the same character and has a similar equation. It is only necessary to suppose that the C of the preceding formulæ is augmented by half the square of the value attributed to the velocity. Thus, in the case of our moon, it is plain the curves of equal velocity will form a series of ovals surrounding the origin, and approaching it, and becoming more nearly circular as the velocity increases.

Applying the simple formulæ, where the solar parallax is neglected, to the moon, we find that the distance of the asymptotic lines, from the origin, is

$$\sqrt{\frac{2C}{3n'^2}} = 500.4992.$$

The distance of the points on the axis of x , at which the moon would remain stationary with respect to the sun, is

$$\sqrt[3]{\frac{\mu}{3n'^2}} = 235.5971.$$

*The corresponding solution, in the more general problem of three bodies, may be seen in the *Mécanique Céleste*, Tom. IV, p. 310.

If the auxillary angle θ is derived from the equation

$$\sin \theta = \frac{9\mu n'}{(2C)^{\frac{3}{2}}}.$$

we get

$$\theta = 32^{\circ} 49' 6''.63;$$

and the distances from the origin, at which the curve of zero velocity intersects the axis of x , are given by the two expressions

$$\frac{2\sqrt{2C}}{3n'} \sin \frac{\theta}{3},$$

$$\frac{2\sqrt{2C}}{3n'} \sin \left(60^{\circ} - \frac{\theta}{3}\right),$$

and the numbers are 109.6772 and 435.5623. These values differ but little from the previous more general determinations.

(To be continued.)

